Quantum Optomechanics: from Gravitational Wave Detectors to Macroscopic Quantum Mechanics

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Contents

1 Overview and Basic Notions 2
   1.1 Overview: Two aspects of quantum measurement 2
   1.2 Origin of the Standard Quantum Limit 3
   1.3 Linear Quantum Measurement Theory 6

2 Various configurations that circumvent the SQL 9
   2.1 Quantization of Light: A Brief Review 9
   2.2 Light Reflection Off a Moving Mirror 12
   2.3 The simplest interferometer 15
   2.4 Surpassing the SQL: frequency-dependent squeezing and homodyne detection 18
   2.5 Speed Meters: modifying the optical transfer function 19
   2.6 Optical Spring: modifying mechanical response 20

3 Towards a more systematic understanding 22
   3.1 The Mizuno Theorem for Interferometers with Free Masses 22
   3.2 White Light Cavities 22
   3.3 Proof of the Mizuno Theorem and the Quantum Cramer-Rao Bound 24
   3.4 Realizing White-light Cavity using Unstable Filters 28
   3.5 Summary 28

4 Quantum State Preparation and Verification 30
   4.1 Zero-Point Fluctuation of an Oscillator and the Fluctuation-Dissipation Theorem 30
   4.2 Stochastic Schroedinger/Master Equations 30
   4.3 Wiener Filter and Conditional Gaussian-State Preparation 31
   4.4 Preparation of non-Gaussian Quantum States 31
   4.5 Quantum-State Tomography 31

5 Testing Quantum Mechanics 32
   5.1 Collapse Models 32
   5.2 Semiclassical Gravity 32
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Overview and Basic Notions

1. Overview: Two aspects of quantum measurement

There are two aspects of quantum measurement that this lecture will cover.

1.1.1 Measurement of a weak force

We can imagine a weak classical force that we need to measure. This force acts on a test mass, creating displacement. Then optical field is used to read out the displacement. In this context, we need to minimize the noise that arise due to quantum fluctuations of both the mass and the light. We will also have to isolate the system from other classical forces, e.g., thermal fluctuations. As we shall see soon in these lectures, the quantum noise level consist of the shot noise of light, which can be viewed as arising from the inaccuracy of repetitive measurements, and the radiation-pressure noise, which enforces the measurement-induced back action. Together, the trade-off of these two types of noise gives rise to the so-called Standard Quantum Limit, which was first realized by Vladimir Braginsky in the 1960s when he was contemplating the ultimate limit of his precision measurement devices that use macroscopic test masses. The SQL is not a limitation for the measurement precision, after all – but it provides the magnitude in which we must consider both measurement precision and measurement-induced back action. Lecture 1, 2 and 3 will be devoted to this thread of thought.

In Lecture 1, we will develop a linear quantum measurement formalism that clarifies in which context the SQL would apply, and see how we might in fact circumvent to the SQL. We will also introduce the necessary notations.

In Lecture 2, we will then present a zoo of configurations that allows the interferometer to circumvent the SQL – we finally realize that optical losses, which destroys quantum coherence, is the major obstacle toward sub-SQL interferometers.

Given all these possibilities, what are the better ones that are most immune to optical losses? Is there a fundamental limit that incorporates losses? These questions are not yet answered, but with these questions in mind, we will move on to the quest for a more systematic understanding of such interferometers, which will be presented in Lecture 3.

1.1.2 Quantum Mechanics of Macroscopic Objects

Alternatively, especially in recent years, we can study a second topic, which is to try to prepare, manipulate and characterize the quantum state of a macroscopic quantum object. As we will show in Lecture 4, the free-mass SQL actually provides a benchmark for the “quantum-ness” of the system. If a device has classical sources of noise below...
the SQL, then these classical noises will not affect the preparation and verification of the quantum state of the test mass. Furthermore, the quantum noise of the device characterizes its efficiency in preparing and probing the quantum state of the test mass. We will then show that a sub-SQL device can be used to prepare nearly pure quantum states, mechanical entanglement — as well as non-Gaussian quantum states that have no classical counterparts.

Finally, in Lecture 5, let us test quantum mechanics. Quantum mechanics contains two parts: (i) unitary evolution for the joint wavefunction of a quantum system, and (ii) Born’s rule that converts the wavefunction into probability distributions that we can observe. Now with macroscopic objects involved in the unitary evolution, we have enlarged the regime of (i). In this way, we can test certain modifications of quantum mechanics when the number of particles involved is very large. Furthermore, we should also formulate tests of Born’s rule.

### 1.2 Origin of the Standard Quantum Limit

The Standard Quantum Limit can be derived from two different ways.

#### 1.2.1 Focusing on optical fluctuations

Let us first motivate the SQL from the optical point of view, by considering the sensing noise and the back-action noise. For optical field trying to detect a displacement $x$, we have

$$\delta \phi = \frac{2 \omega_0 x}{c} \quad (1.1)$$

where $\omega_0$ is the angular frequency of light. For $N$ photons, we will have

$$\delta \phi \sim \sqrt{\frac{1}{N}}, \quad (1.2)$$

which arises from the discreteness of photons; hence,

$$\delta x \sim \frac{c}{2 \omega_0} \sqrt{\frac{1}{N}} \quad (1.3)$$

For a duration of $\tau$,

$$N = \frac{P}{\hbar \omega_0} \tau \quad (1.4)$$

where $\omega_0$ is the angular frequency of light, we have

$$\delta x \sim \frac{c}{2 \omega_0} \sqrt{\frac{\hbar \omega_0}{P}} \sqrt{\frac{1}{\tau}} \quad (1.5)$$

Note that for a random process $A$ with a white spectrum $S_A$, if we measure it for a duration of $\tau$, taking the time average, then the standard deviation of that mean is given by

$$\delta A_{\tau} \approx \sqrt{\frac{S_A}{\tau}} \quad (1.6)$$
From this, we can extract a steady-state sensing noise spectrum of

\[ S_x^{\text{sens}} \sim \frac{c^2 \hbar}{4 \omega_0 P} \]  

In a similar fashion, we can also derive the back-action noise. For duration of \( \tau \), the uncertainty of the number of photons hitting the mirror is given by

\[ \delta N \sim \sqrt{N} = \sqrt{\frac{P}{\hbar \omega_0 \tau}} \]  

this means the force acting on the mirror has an uncertainty of

\[ \delta F = \frac{2 \hbar \omega_0}{c} \delta N \sim \frac{2}{c} \sqrt{\frac{\hbar \omega_0 P}{\tau}} \]  

this corresponds to a spectrum of

\[ S_F \sim \frac{4 \hbar \omega_0 P}{c^2} \]  

which causes a back action noise of

\[ S^{\text{BA}}_x = \frac{1}{M^2 \Omega^4} \frac{4 \hbar \omega_0 P}{c^2} \]  

If we superimpose these two types of noise, we obtain

\[ S_x^{\text{tot}} = S_x^{\text{sens}} + S^{\text{BA}}_x = \frac{c^2 \hbar}{4 \omega_0 P} + \frac{1}{M^2 \Omega^4} \frac{4 \hbar \omega_0 P}{c^2} \geq \frac{2 \hbar}{M \Omega^2} \]  

This turns out to be equal to the free-mass Standard Quantum Limit. Note that the SQL is at the “same scale” as the zero-point fluctuation of a harmonic oscillator with eigenfrequency \( \Omega \).

1.2.2 Focusing on mechanical quantum state

We can also focus on the test mass. Suppose, that we would like to measure the position of the mass successively at different moments of time. For this, let us consider the Heisenberg Operator \( \hat{x}_H(t) \) of a free mass

\[ \hat{x}_H(t) = \hat{x}_0 + \frac{\hat{p}_0 t}{M} \]  

We also note that

\[ [x_H(t), x_H(t')] = \frac{i \hbar (t' - t)}{M}. \]  

These Heisenberg Operators do not commute. We cannot “simultaneously measure” these operators! Well, for \( t \) and \( t' \), they aren’t simultaneous anyway — but what we really mean is that we cannot prepare the test mass into a special quantum state
which has very small quantum uncertainties at both \( t \) and \( t' \). In fact, for any test-mass quantum state,
\[
\Delta x(t) \cdot \Delta x(t') \geq \frac{\hbar}{2M} |t' - t|
\] (1.15)

If we measure with an interval of \( \tau \), and keep \( \Delta x \) the same at all times, we will obtain
\[
\Delta x = \sqrt{\frac{\hbar \tau}{2M}}
\] (1.16)

which is consistent with the SQL.

1.2.3 Quantum Non-Demolition (QND)

As we see from the above, both the two ways have some loopholes. In the first approach, we did not consider the intrinsic dynamics of the test mass, but only the dynamics as driven by light; we did not consider possible correlations between the sensing and the back-action noise, either. We will quickly fix the above problems, and arrive at a formalism that will allow us to compute the quantum noise correctly — and find ways to surpass the SQL.

The second approach was adopted earlier in history. It does not seem to be the most efficient way to obtain a computational tool for the noise spectrum, but this argument highlights the problem: if we are to “measure” non-commuting observables successively, we must add noise. There are two ways to see why “measuring non-commuting observables” is not only tricky but in fact does not have a complete meaning.

First, from an information point of view. If we have \( \hat{x}(t) \), with \( [\hat{x}(t), \hat{x}(t')] \neq 0 \), and a quantum state of the system \( |\psi\rangle \), we do not have a unique way to obtain the probability distribution for \( \tilde{x}(t) \), the measurement record. In fact, we cannot transcribe \( \hat{x}(t) \) for different values of \( t \) into different memory units. By contrast, if we had \( \hat{Z}(t) \) with \( [\hat{Z}(t), \hat{Z}(t')] = 0 \), we can simply project the \(|\psi\rangle\) into simultaneous eigenstates of \( \{\hat{Z}(t)\} \), and that will provide us with the probability distribution for the measuring record. In this way, the hand-off from quantum to classical cannot be done at the level of \( \hat{x}(t) \).

Second, from the dynamical point of view. If we couple \( x \) to another quantum system for “further processing”, e.g., via the interaction Hamiltonian of
\[
V = -xf
\] (1.17)

we will have, for a linear system [or perturbatively for any system]
\[
\hat{x}^{(1)}(t) = \hat{x}^{(0)}(t) + \frac{i}{\hbar} \int_0^t dt' [\hat{x}(t), \hat{x}(t')] f(t')
\] (1.18)

This means, the evolution of the actual \( \hat{x}(t) \), after being coupled to the measuring device, depends on details of the device — unless we specify the details of the device, we cannot obtain a reliable description of the measurement process. From the Schroedinger picture, this also means even if we prepare an initial state for the probe, this state will be “demolished” by the measuring device, in a way that depends on the details of the device.
Historically, one way to circumvent the above problem, was to construct and measure commuting observables, or Quantum Non-Demolition (QND) observables, of the test mass. For a free test mass, this will be

\[ \hat{x} - \frac{\hat{p}}{m}, \quad \hat{p} \]  

(1.19)

and for oscillators, the quadrature operators

\[ \hat{x} \cos \omega_m t - \frac{\hat{p}}{m} \sin \omega_m t, \quad \hat{x} \sin \omega_m t + \frac{\hat{p}}{m} \cos \omega_m t. \]  

(1.20)

The latter has recently been realized by Keith Schwarb’s research group. In addition, since

\[ [\hat{x}(t), \hat{x}(t')] = i\hbar \sin[\omega_m (t' - t)] \]  

(1.21)

one can perform so-called Stroboscopic experiments, at moments of time separated by half period of oscillation. This same strategy has been used in “Pulsed Optomechanics”. These will be excellent ways to quantify the quantum state of the test mass, therefore recently find many applications to the manipulation of the test mass. However, as we shall see later, this is not the only way toward improvement of sensitivity.

The full treatment of the problem will always be that the test mass and the light together form a system that is being measured. For this system, we measure the out-going field at different moments of time, which automatically commute, and are trivially QND observables. This doesn’t mean we don’t have noise — it just means there does not need to be an additional noise than those already in the optomechanical quantum state.

### 1.3 Linear Quantum Measurement Theory

Let us develop the quantum measurement theory more precisely, following Braginsky and Khalili.

#### 1.3.1 Equation of Motion

Suppose now we have a system that consists the probe (test mass) and the device (optical field). We have the Hamiltonian of

\[ H = H_P + H_D - xF - xG \]  

(1.22)
where \( x \) belongs to the probe (test mass), and \( F \) belongs to the device (light). Here \( G \) is the classical force that we would like to measure. We can write

\[
Z^{(1)}(t) = Z^{(0)}(t) + \frac{i}{\hbar} \int_0^t dt' C_{ZF}(t-t')x^{(1)}(t'). \quad (1.23)
\]

\[
x^{(1)}(t) = x^{(0)}(t) + \frac{i}{\hbar} \int_0^t dt' C_{xx}(t-t') \left[ G(t') + F^{(1)}(t') \right]. \quad (1.24)
\]

\[
F^{(1)}(t) = F^{(0)}(t) + \frac{i}{\hbar} \int_0^t dt' C_{FF}(t-t')x^{(1)}(t'). \quad (1.25)
\]

Here we have defined

\[
C_{AB}(t-t') = \left[ \hat{A}(t), \hat{B}(t') \right] \quad (1.26)
\]

We suppose that \( Z(t) \) is an out-going field that can be measured, with

\[
[Z(t), Z(t')] = 0 \quad (1.27)
\]

Let us consider the special case where

\[
C_{ZF} = i\hbar \delta(t-t') \quad (1.28)
\]

so that the out-going field has a nearly instantaneous response to the motion of the test mass. Let us also assume that

\[
C_{FF}(t-t') = [F(t), F(t')] = 0 \quad (1.29)
\]

This is as if we have a series of harmonic oscillators that interact with the test mass. In this case, we find

\[
Z^{(1)}(t) = Z^{(0)}(t) + x^{(1)}(t), \quad (1.30)
\]

\[
x^{(1)}(t) = x^{(0)}(t) + \frac{i}{\hbar} \int_0^t dt' C_{xx}(t-t')(t') \left[ G(t') + F^{(0)}(t') \right]. \quad (1.31)
\]

or

\[
Z^{(1)}(t) = Z^{(0)}(t) + x^{(0)}(t) + \frac{i}{\hbar} \int_0^t dt' C_{xx}(t-t')F^{(0)}(t')
\]

\[
+ \frac{i}{\hbar} \int_0^t dt' C_{xx}(t-t')G(t'). \quad (1.32)
\]

### 1.3.2 The Heisenberg Uncertainty and the Standard Quantum Limit

Relations (1.27)–(1.29) gives rise to the Heisenberg Uncertainty Relation in the spectral domain:

\[
S_{ZZ}S_{FF} - |S_{ZF}|^2 \geq \hbar^2 \quad (1.33)
\]

If we calculate the noise spectrum, we will have

\[
S_x = S_x^p + S_{ZZ} + 2Re [R_{xx}S_{ZF}] + |R_{xx}|^2 S_{FF}. \quad (1.34)
\]
Here we have defined

$$ R_{xx}(\Omega) = \frac{i}{\hbar} \int_{0}^{+\infty} C_{xx}(t)e^{i\Omega t} dt $$

We also have a definition of spectrum that is slightly different from the one by Clerk et al. We have a symmetric spectrum:

$$ \frac{1}{2} S_{AB}(\Omega) 2\pi \delta(\Omega - \Omega') = \langle A(\Omega)B^\dagger(\Omega') + B^\dagger(\Omega')A(\Omega) \rangle $$

For a mechanical oscillator with eigenfrequency $\omega_m$, we have

$$ R_{xx} = -\frac{1}{m(\omega^2 - \omega_m^2)} $$

The zero-point part is focused near the eigenfrequency of the oscillator. If we measure off-resonance, then we will only be limited by the optical noises. However, the optical noise does bear some imprint from the dynamics of the test mass. In the case when $S_{ZF} = 0$, we will have

$$ S_x = S_{ZZ} + |R_{xx}|^2 S_{FF} \geq 2\hbar|R_{xx}| $$

Or

$$ S_{SQL}^x = 2\hbar|R_{xx}| = \frac{2\hbar}{m|\omega^2 - \omega_m^2|} $$

We also have, for force measurement

$$ S_{SQL}^G = \frac{2\hbar}{|R_{xx}|} = 2\hbar m|\omega^2 - \omega_m^2| $$

As we can see here, if we are near the resonance of a harmonic oscillator, the SQL is particularly low. However, there we will have to be careful about the zero-point fluctuation, as well as thermal noise, in practice. That we will postpone till lecture 4, when we deal with the quantum state of the oscillator. In the rest of the lectures, we will mainly focus on the fluctuations of the light — away from the oscillator’s eigenfrequency. As we see from the above, if we build the appropriate correlations between $Z$ and $F$, then the SQL is not enforced. In other words, the test mass is only one degree of freedom, while the light has an infinite number of degrees of freedom. In the frequency domain, all the quantum-ness of the test mass is concentrated near its eigenfrequency — when we are away from that, we will only have to deal with fluctuations of light.

To see how it can be done in optomechanical devices, we will have to first review some basics of quantum optics.
Various configurations that circumvent the SQL

In this lecture, I will discuss various ways the SQL can be surpassed in GW detectors. With this, we will illustrate concepts such as back-action evasion and the optical spring effect. These are meant to provide examples of what could be done.

2.1 Quantization of Light: A Brief Review

In this chapter, I will discuss the basics of field quantization.

2.1.1 Field operators

For a field propagating along the $x$ direction with speed $c$, we write, in the Heisenberg Picture,

$$\hat{E}(t,x) = \int_0^{+\infty} \frac{d\omega}{2\pi} \sqrt{\frac{2\pi \hbar \omega}{Ac}} \left[ a_\omega e^{ikx-i\omega t} + a_\omega^\dagger e^{-ikx+i\omega t} \right]$$

(2.1)

For simplicity, here we have taken a transverse mode that is unity within a cylinder with area $A$, and zero amplitude outside. Here we note that $a_\omega^\dagger$ is the creation operator for a spatial mode whose wave vector is given by $k = \omega/c$, with a spatial wavefunction of $e^{ikx}$. The creation and annihilation operators satisfy

$$[\hat{a}_\omega, \hat{a}_\omega^\dagger] = 2\pi \delta(\omega - \omega'), [\hat{a}_\omega, \hat{a}_\omega] = [\hat{a}_\omega^\dagger, \hat{a}_\omega^\dagger] = 0.$$  

(2.2)

with all other operators vanishing. The normalization here follows a Gaussian unit system, with energy density given by $E^2/(4\pi)$.

If we study phenomenon around a central frequency $\omega_0$, we will be focusing on operators $a_{\omega_0\pm\Omega}$ and $a_{\omega_0\pm\Omega}^\dagger$. It is often convenient to re-organize the operators into quadrature operators:

$$\hat{a}_1(\Omega) = \frac{a_{\omega_0+\Omega} + a_{\omega_0-\Omega}^\dagger}{\sqrt{2}}, \quad \hat{a}_2(\Omega) = \frac{a_{\omega_0+\Omega} - a_{\omega_0-\Omega}^\dagger}{\sqrt{2}}.$$ 

(2.3)

We can also define, in the time domain,

$$\hat{a}_{1,2}(t) = \int_0^\Lambda \frac{d\Omega}{2\pi} \left[ \hat{a}_{1,2}(\Omega)e^{-i\Omega t} + \hat{a}_{1,2}^\dagger(\Omega)e^{i\Omega t} \right],$$

(2.4)

where $\Lambda$ is a cut-off that is much less than $\omega_0$ but much higher than the bandwidth of the process we care about. In this way, field around $\omega_0$ can be written as
Various configurations that circumvent the SQL

\[ \hat{E} = \hat{a}_1(t) \cos \omega_0 t + \hat{a}_2(t) \cos \omega_0 t. \]

In general, we can define the \( \theta \)-quadrature as

\[ a_\theta = a_1 \cos \theta + a_2 \sin \theta. \]

This can be done either in the time domain or in the frequency domain.

### 2.1.2 Gaussian States

The simplest quantum states for light are Gaussian states — those whose Wigner functions are Gaussian. The vacuum state is a Gaussian pure state. The vacuum state satisfies

\[ \hat{a}_\omega |0\rangle = 0. \]

Using the definition of quadratures and the commutation relations, we can show that in a vacuum state, we have

\[ S_{a_1 a_1} = S_{a_2 a_2} = 1, \quad S_{a_1 a_2} = 0. \]

Other Gaussian pure states can be formed through displacement and squeezing operators. The displacement operator is

\[ D[\alpha(\omega)] = \exp \left\{ \int \frac{d\omega}{2\pi} \left[ \alpha(\omega) a_\omega^\dagger - \alpha^*(\omega) a_\omega \right] \right\} \]

while the squeezing operator is

\[ S[\beta(\omega_1, \omega_2)] = \exp \left\{ \int \frac{d\omega_1}{2\pi} \int \frac{d\omega_2}{2\pi} \left[ \beta(\omega_1, \omega_2) a_{\omega_1} a_{\omega_2}^\dagger - \beta^*(\omega_1, \omega_2) a_{\omega_1}^\dagger a_{\omega_2} \right] \right\} \]

Both displacement and squeezing operators are unitary. It is possible to always return to the vacuum state, but transform the field operators. In coherent states, we simply add coherent amplitude to the field operators,

\[ \hat{a}_\omega \to \alpha(\omega) + \hat{a}_\omega \]

while for squeezed state, it is possible to write

\[ \hat{a}_\omega \to \int \frac{d\omega'}{2\pi} \left[ \mu(\omega, \omega') \hat{a}_{\omega'} + \nu(\omega, \omega') \hat{a}_{\omega'}^\dagger \right] \]

These are called Bogolubov transformations. Valid Bogolubov transformations are the one that preserve the canonical commutation relations.
2.1.3 Special Coherent and Squeezed States in the Quadrature Representation

We shall consider a special coherent state, which, after the unitary transformation back to vacuum, has

\[
E(t, x) = \sqrt{\frac{4\pi\hbar\omega_0}{Ac}} \left[ \left( \sqrt{\frac{2P}{\hbar\omega_0}} + a_1(t) \right) \cos \omega_0 t + a_2(t) \sin \omega_0 t \right]
\]  

(2.13)

This will be a light beam with power \(P\). This is often referred to as the carrier field. Note that \(a_1\) and \(a_2\) are now amplitude and phase modulations to the carrier. In this way, sometimes they are referred to as the amplitude and phase quadratures.

We shall consider special squeezed states that corresponds to transformations like

\[
a_\theta \rightarrow e^{+\xi}a_\phi, \quad a_{\phi+\pi/2} \rightarrow e^{-\xi}a_{\pi/2+\phi},
\]  

(2.14)

This can be done either in the time domain or in the frequency domain. In the next lecture, we will use a particular type of squeezed states, with

\[
S = \exp \left[ \int_0^\infty \frac{d\Omega}{2\pi} \left( \chi_\Omega \hat{a}^\dagger_{\omega_0+\Omega} \hat{a}^\dagger_{\omega_0-\Omega} - \chi_\Omega^* \hat{a}_{\omega_0+\Omega} \hat{a}_{\omega_0-\Omega} \right) \right]
\]  

(2.15)

By defining \(\chi_\Omega = \xi_\Omega e^{-2i\phi_\Omega}\) we have a frequency dependent squeezing of the quadratures:

\[
\begin{pmatrix}
a_1 \\
a_2
\end{pmatrix} = \begin{pmatrix}
\cosh \xi + \sinh \xi \cos 2\phi & -\sinh \xi \sin 2\phi \\
-\sinh \xi \sin 2\phi & \cosh \xi - \sinh \xi \cos 2\phi
\end{pmatrix} \begin{pmatrix}
a_1 \\
a_2
\end{pmatrix}
\]  

(2.16)

On the other hand, \(S\) generates pairs of photons, it is not hard to show that each pair has a relative time separation that is given by the inverse Fourier transform of \(\chi\). In this way, if \(\chi\) is to vary within a narrow bandwidth, this means the two photons are separated by a long time. If we detect one of those photons, then we will be heralding a photon with a long spatial mode.
2.2 Light Reflection Off a Moving Mirror

Let us now consider the coupling between light and mirror. Suppose the incident beam is reflected off a mirror which has a displacement of $X(t)$. Suppose $X$ is much less than the wavelength of light, and that $X$ is much less than the speed of light, we can write

$$E_{\text{out}}(t) = E_{\text{in}}[t - 2X(t)/c] \tag{2.17}$$

We shall consider two special cases where we choose different linearizations. It is perhaps instructive to consider the following Hamiltonian for a single cavity mode coupled to a movable mirror:

$$V = -a^\dagger a x \tag{2.18}$$

We can either write

$$a \rightarrow \langle a \rangle + \delta a \tag{2.19}$$

and expand in $\delta a$ and $x$, or we can write

$$x \rightarrow \langle x \rangle \tag{2.20}$$

and throw away its quantum fluctuations.
2.2.1 Strong Carrier, Small motion: Sensing of Motion

Let us write, in the quadrature representation

\[
E_{\text{in}}(t, x) = \sqrt{\frac{4\pi\hbar\omega_0}{Ac}} \left[ \left( \sqrt{\frac{2P}{\hbar\omega_0}} + a_1(t) \right) \cos\omega_0 t + a_2(t) \sin\omega_0 t \right]
\]  \hspace{1cm} (2.21)

and

\[
E_{\text{out}}(t, x) = \sqrt{\frac{4\pi\hbar\omega_0}{Ac}} \left[ \left( \sqrt{\frac{2P}{\hbar\omega_0}} + b_1(t) \right) \cos\omega_0 t + b_2(t) \sin\omega_0 t \right]
\]  \hspace{1cm} (2.22)

In this case, we expand the \(t\) inside \(\cos\) and \(\sin\), and obtain

\[
b_1(t) = a_1(t), \quad b_2(t) = a_2(t) + \frac{2\omega_0 X(t)}{c} \sqrt{\frac{2P}{\hbar\omega_0}}
\]  \hspace{1cm} (2.23)

Now, if we detect \(b_2\), we have a shot noise of

\[
S_Z = \frac{c^2 h}{8\omega_0 P}
\]  \hspace{1cm} (2.24)

We can also compute the back-action force spectrum onto the mirror, which is given by

\[
F = \frac{2AE^2}{c^2 4\pi} = \frac{2\hbar\omega_0}{c^2} \sqrt{\frac{2P}{\hbar\omega_0}} a_1
\]  \hspace{1cm} (2.25)

This gives

\[
S_F = \frac{8\hbar\omega_0 P}{c^2}
\]  \hspace{1cm} (2.26)

We also have \(S_{FZ} = 0\). This recovers the result at the beginning of the lecture. However, we do see that \(S_{FZ}\) can be non-zero, and this will be used in the next lecture for
Various configurations that circumvent the SQL

2.2.2 No Carrier, Big Motion: Dynamical Casimir Effect

It is actually also interesting to consider the case without carrier, but the motion is not very small (but still smaller than the wavelength of light). In fact, let us suppose

\[ X(t) = X_0 \cos(2\omega_0 t) \tag{2.27} \]

We can write

\[
E_{\text{out}}(t) = \int \frac{d\omega}{2\pi} \sqrt{\omega} a_\omega e^{-i\omega(t-2X_0/c)} + \text{h.c.}
\]

\[
= \int \frac{d\omega}{2\pi} \sqrt{\omega} \left[ a_\omega e^{-i\omega t} + i \frac{\omega X_0}{2c} a_\omega e^{-i(\omega+2\omega_0)t} + i \frac{\omega X_0}{2c} a_\omega e^{-i(\omega-2\omega_0)t} \right]
\]

\[
+ \int \frac{d\omega}{2\pi} \sqrt{\omega} \left[ a_\omega^\dagger e^{+i\omega t} - i \frac{\omega X_0}{2c} a_\omega^\dagger e^{+i(\omega+2\omega_0)t} - i \frac{\omega X_0}{2c} a_\omega^\dagger e^{+i(\omega-2\omega_0)t} \right] \tag{2.28}
\]

Here we see a conversion between different frequencies. However, in QFT, conversion between positive and negative frequencies have the profound consequence of particle creation! This takes place for all frequencies less than \(2\omega_0\).

If we restrict ourselves to the \(\omega_0 + \Omega\) component of the out-going field, with \(\Omega \ll \omega_0\), we can write

\[
b_{\omega_0+\Omega} = a_{\omega_0+\Omega} + \sqrt{\frac{\omega_0 - \Omega}{\omega_0 + \Omega}} \left[ -i(\omega_0 - \Omega) X_0 \right] a_{\omega_0-\Omega} = a_{\omega_0+\Omega} - \frac{i \omega_0 X_0}{2c} a_{\omega_0-\Omega} \tag{2.29}
\]

This means

\[
b_{\pi/4} = \left( 1 - \frac{\omega_0 X_0}{2c} \right) a_{\pi/4}, \quad b_{3\pi/4} = \left( 1 + \frac{\omega_0 X_0}{2c} \right) a_{3\pi/4} \tag{2.30}
\]

This is squeezing! The squeeze factor is

\[
r = \frac{\omega_0 X_0}{2c} = \frac{V_0}{2c} \tag{2.31}
\]

In other words: as we shake the mirror at \(2\omega_0\), pairs of photons will be created.

This is the Dynamical Casimir Effect: as we shake anything that has a non-zero polarizability, it will radiate photon pairs. Another way to understand this is the following: any object with non-zero polarizability couples to the surrounding EM field, so that the state of the surrounding EM field is not exactly at vacuum, but, instead, polarized by that object. As we shake that object, these polarization photons will escape, causing the dynamical Casimir effect.

In reality, it is perhaps very difficult to shake the mirror fast enough in order to detect the photons. However, one way to achieve a similar effect is to use a non-linear crystal with \(\chi^{(2)}\) nonlinearity. Suppose we have a second harmonic of our carrier frequency \(\omega_0\), driving a crystal with \(\chi^{(2)}\) nonlinearity, its refractive index will be modulated at \(2\omega_0\),

\[
n = n_0 + n_1 E \cos 2\omega_0 t \tag{2.32}
\]

this creates the same effect of a mirror moving at \(2\omega_0\), producing squeezed vacuum.
2.2.3 Extensions of the Dynamical Casimir Effect

The fundamental reason for particle creation that we discussed above is a non-uniform map in time. Any such process will cause particle creation. These types of processes take place in the early universe, when particles are created during inflation, they also take place during gravitational collapse. The collapse of a star into a black hole distorts the space-time geometry in an ultimate way, when a boundary forms between those rays that escape toward infinity and those that do not escape. The Dynamical Casimir Effect in this case gives the Hawking Radiation.

2.3 The simplest interferometer

2.3.1 Single Movable Mirror

For a single test mass, we simply need to express $X$ in terms of the back-action force. In the frequency domain, we can write the quadrature input-output relation as

\begin{align}
    b_1 &= a_1 \\
    b_2 &= a_2 - \kappa a_1 + \sqrt{2\kappa} \frac{x_{GW}}{x_{SQL}}
\end{align}

(2.33)

(2.34)

here $x_{GW}$ is the signal,

\[ x_{SQL} \equiv \sqrt{S_2^{SQL}} = \sqrt{\frac{2h}{m\Omega^2}} \]

(2.35)

and

\[ \kappa = \frac{4\omega_0 P}{mc^2\Omega^2} \]

(2.36)
Various configurations that circumvent the SQL

Fig. 2.6 Phasor Diagram illustrating the input-output relation.

We can use the following diagram to illustrate the situation. Basically, the signal is within the phase quadrature — but in that quadrature we also have the back-action noise. For a vacuum input state, we can show that the sensing and back-action noise together enforces the free-mass SQL:

$$S_x = \left[ \frac{1}{\kappa} + \kappa \right] \frac{S_{SQL}^x}{2} \geq S_{SQL}^x$$

However, already from here, we see how to circumvent the SQL. If we squeeze a combination of $a_2 - \kappa a_1$, we will be able to suppress the entire noise by the squeeze factor. If we detect a quadrature other than the phase quadrature, we can also allow the back-action to partially cancel, resulting in noise below the SQL. In both cases, we will be having shot noise correlated with radiation-pressure noise. The challenge, it seems, to be that $\kappa$ is frequency dependent — hence we will need to have frequency dependent squeezed vacuum (as mentioned in the previous lecture) and/or frequency dependent homodyne detection.

2.3.2 Fabry-Perot Michelson Interferometers

Before we go into the details of how to realize such frequency dependence, let us present a more realistic configuration, that involves interferometers. First of all, we have a Michelson configuration that is at the dark port, in order for classical laser noise to cancel. In this way, the common mode of the interferometer provides the driving, while the differential mode is where the quantum dynamics takes place. When injecting squeezed vacuum, it will be into the dark port. The signal also emerges from the dark port; it can be detected via homodyne or heterodyne detection. We shall assume the former, which basically measures one particular quadrature. If we superimpose the out-going field with a local oscillator at the $\theta$-quadrature, and then detect the intensity of light, we will obtain $b_\theta$.

Another improvement that we need is resonant cavities in the arm, plus the power recycling cavity (which Ron Drever invented at Les Houches in 1983). Ingredients for
obtaining the input-output relation involves the input-output relations of: (i) movable
mirror, as already given above, (ii) propagation through free space for an \( L \) distance,

\[
\begin{bmatrix}
    b_1(\Omega) \\
    b_2(\Omega)
\end{bmatrix}
= e^{i\Omega L/c} \begin{pmatrix}
    \cos \phi & \sin \phi \\
    -\sin \phi & \cos \phi
\end{pmatrix}
\begin{bmatrix}
    a_1(\Omega) \\
    a_2(\Omega)
\end{bmatrix}
\]

(2.38)

where \( \phi = \omega_0 L/c \), and (iii) transmission and reflection at a mirror:

\[
d_j = \sqrt{R}a_j + \sqrt{T}b_j, \quad c_j = \sqrt{T}a_j - \sqrt{R}b_j.
\]

(2.39)

The input output relation for a Fabry-Perot Michelson interferometer to gravita-
tional wave with amplitude \( h \) is given by

\[
b_1 = e^{2i\beta}a_1
\]

(2.40)

\[
b_2 = e^{2i\beta}[a_2 - Ka_1] + e^{i\beta}\sqrt{2K}\frac{h_{GW}}{h_{SQL}}
\]

(2.41)

Here

\[
\beta = \arctan \frac{\Omega}{\gamma}
\]

(2.42)

with

\[
\gamma = Tc/(4L)
\]

(2.43)

the cavity bandwidth, and

\[
K = \frac{2\gamma \Theta^3}{\Omega^2(\Omega^2 + \gamma^2)} , \quad \Theta^3 = \frac{8\omega_0 P_c}{mLc},
\]

(2.44)
where $P_c$ is the power circulating in the arms. We also have

$$h_{\text{SQL}} = \sqrt{\frac{8h}{m\Omega^2 L^2}}$$

(2.45)

This is after accounting for four equal test masses, and response to the GW. The structure of this input-output relation is the same as before — just with a more complex $\mathcal{K}$, which contains not only the poles of a free mass, but also that of the cavity.

### 2.4 Surpassing the SQL: frequency-dependent squeezing and homodyne detection

#### 2.4.1 The required frequency dependence and the optical filters

It is not hard to work out the required frequency dependence. For input squeezing, suppose we need a device that applies rotation $\phi(\Omega)$ to the input fields, with

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} e^{-r} \\ e^{+r} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

(2.46)

In order for $a_2 - \kappa a_1$ to be squeezed, we need to require

$$\tan \phi_{\text{squ}} = 1/\kappa$$

(2.47)

At low frequencies, we are squeezing $a_1$, while at high frequencies, we are squeezing $a_2$.

For frequency-dependent homodyne detection, we first apply frequency-dependent quadrature rotation to the out-going field, and then make a frequency-independent homodyne detection of the first quadrature:

$$\begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

(2.48)

This requires

$$\tan \phi_{\text{var}} = 1/\kappa$$

(2.49)

as well. This is called the variational readout scheme.

It is also possible to show that such a rotation can be achieved with detuned Fabry-Perot cavities. In fact, we can show that a series of $n$ cavities can always achieve rotation with the following form:

$$\tan \phi = \frac{\sum_{j=0}^{n} a_j \Omega^{2j}}{\sum_{j=0}^{n} b_j \Omega^{2j}}, \quad a_n b_n \neq 0$$

(2.50)
2.4.2 Role of optical losses

The above schemes are in principle able to allow the interferometer to evade back action, and improve sensitivity indefinitely. The limitation will be optical losses. Schemes that are mathematically equivalent without losses could be very different when losses are considered.

First of all, in order to achieve frequency dependence, the filter cavities must have a bandwidth that is comparable to the detection bandwidth — which for GW detection will be below kHz. Second, in particular for the variational readout scheme, the back-action evading quadrature has very little signal, increasing its susceptibility to losses.

In the variational readout scheme, sensitivity of the interferometer will be highly contaminated by loss. Farid Khalili derived a SQL-beating limit due to optical losses, and it is given by

\[ \sqrt{S_h} \geq (e^{-2r} \epsilon)^{1/4} \sqrt{S^{\text{SQL}}_h} \]  

(2.51)

where \( \epsilon \) is the loss in power, and \( e^{-2r} \) is the power squeeze factor that is injected into the interferometer. For 10 dB squeezing and 1% loss, we can only hope to beat the SQL by 5. However, in the case when \( \mathcal{K} \) is too large, the beating factor of 5 is only a maximum — performance at lower frequencies tend to be much worse.

2.5 Speed Meters: modifying the optical transfer function

2.5.1 Velocity versus Momentum

The speed meters are another strategy. The motivation was that momentum is a QND observable — but its advantage really lies in other places.
Momentum is a conserved quantity of a free mass, hence a QND observable. However, let us note that once we couple the velocity of a mass to an external field, there will be the distinction between canonical and kinetic momentum.

\[
L = \frac{m\dot{x}^2}{2} \Rightarrow p = m\dot{x}.
\]

(2.52)

\[
L = \frac{m\dot{x}^2}{2} - \alpha \dot{x}a_1 \Rightarrow p = m\dot{x} - \alpha a_1.
\]

(2.53)

This means, even though canonical momentum is still conserved, it will not be velocity; there is back-action on the velocity! However, there could still be some practical advantages, though.

### 2.5.2 Optical Speed Meters

In order for the interferometer to measure the speed of the mirror, we can either add an additional cavity into the dark port, or build a Sagnac interferometer. There is in fact a mathematical mapping between these two types of interferometers, because in both cases we have two optical oscillators coupled with one free mass. In this case, we can achieve

\[
\mathcal{K} = \frac{2\gamma \Theta^3}{(\Omega^2 - \omega_s^2)^2 + \gamma^2\Omega^2}
\]

(2.54)

At low frequencies, \( \mathcal{K} \) is a constant, and this represents a response to the velocity of the mass. Having this \( \mathcal{K} \) approximately constant allows the interferometer to already beat the SQL without variational readout or frequency-dependent squeezing, although mathematically the optimal will be to use both schemes above. However, if the same level of loss and squeezing can be achieved, the \( \mathcal{K} \) for a speed meter is more advantageous when it comes to the susceptibility to losses.

### 2.6 Optical Spring: modifying mechanical response

In general, we have \([F(t), F(t')] \neq 0\). This means the optical field can modify the dynamics of the test mass. In fact, if \( F \) responds to the motion of the mirror, this is really a spring constant. This effect is familiar to this audience.

#### 2.6.1 Optical Transfer Function

Historically, in GW detection, signal recycling technique was introduced to modify the optical transfer function — so that GW with non-zero frequency is detected with
more sensitivity. One can simply view the system as a single detuned cavity, with bandwidth $\gamma$ and detuning frequency $\Delta$, or $\omega_0 = \omega_c + \Delta$. Assuming that the mirrors do not move under radiation pressure, this type of configuration will have a sensitivity that is peaked at $\Omega = \Delta$, with a width of $\gamma$.

2.6.2 Dynamics

As power is increased, as it turns out, the dynamics of the mirror is modified. As we compute $[F(t), F(t')]$, we obtain the spring constant,

$$K = -\frac{\Theta^3 \Delta}{(\Omega + i\gamma)^2 - \Delta^2},$$

(2.55)

Near $\Omega \approx 0$, we can always expand

$$K(\Omega) = K_{\text{opt}} - i\gamma_{\text{opt}} \Omega,$$

(2.56)

where $K_{\text{opt}}$ is spring constant at DC, and $\gamma_{\text{opt}}$ is optomechanical damping.

For the “blue detuned” case, $\Delta > 0$, we have

$$K_{\text{opt}} > 0, \quad \gamma_{\text{opt}} < 0.$$  

(2.57)

this is a restoring force with anti-damping. For the “red detuned” case, $\Delta < 0$, we have

$$K_{\text{opt}} < 0, \quad \gamma_{\text{opt}} > 0.$$  

(2.58)

this is a restoring force with anti-damping. This means one cannot make a stable optical trap for a free mass with a single optical spring, but can with two.

2.6.3 Quantum Noise spectrum

It is straightforward to compute the noise spectrum in the frequency domain. One simply combines the input-output relation of a single mirror with a detuned cavity. Even if the system is unstable, one can show that if we stabilize its dynamics using feedback, we can still achieve the same sensitivity as calculated in the frequency domain.

For blue detuned case, the noise spectrum has a dip near the “optomechanical resonance”, which is the shifted mirror eigenfrequency. In the red detuned case, the noise spectrum does not have a dip. One can resort to the argument that since we have an oscillator with eigenfrequency in the detection band, sensitivity around that frequency is improved.
3
Towards a more systematic understanding

Having seen the previous lecture, we might wonder: with all these configurations, what do we need to do? How does optical spring fit into the picture?

3.1 The Mizuno Theorem for Interferometers with Free Masses

Let us summarize our understanding here. It seems that without back-action, or optical spring — or in cases with back-action evasion, we have a clear picture. In fact, here I quote the so-called Mizuno Theorem, which states that

$$\int \frac{d\Omega}{2\pi} \frac{1}{S_h(\Omega)} \propto PL \propto \mathcal{E}, \quad (3.1)$$

that is, this quantity is independent from the optical bandwidth, or detuning; it is simply proportional to the energy stored in the cavity. In fact, if we add squeezing, we might write

$$\int \frac{d\Omega}{2\pi} \frac{1}{S_h(\Omega)} \propto \mathcal{E} e^{2r}, \quad (3.2)$$

where the squeeze factor is added by hand. As one can check, all configurations that test-mass dynamics remain unchanged satisfy this relation. [In cases with optical springs, Haixing Miao has argued that it also applies, if we change $e^{2r}$ into the ponderomotive squeezing due to the optical spring.]

This is also called the “Energetic Quantum Limit”. In fact, in this lecture, we shall show that this is intimately related to the Quantum Cramer-Rao bound. Here we can also see that the figure of merit above is the interferometer’s SNR for a pulse with zero width.

3.2 White Light Cavities

The Mizuno theorem was proposed many years ago by J. Mizuno when he was studying the general features of signal recycling interferometers. Why do we revive this again? This was because we were trying to disprove a seemingly crazy strategy for improving detector sensitivity, namely white-light cavities. These was proposed in the 1990s using atomic medium, then using gratings, and later again by Shahriar et al. using atomic medium. For example, if we have two gain peaks separated in the spectrum, then for a carrier half way between the peaks, we can construct a phase-insensitive regime in which we have anomalous dispersion.
3.2.1 What are white light cavities? Why are they good?

White-light cavities are the ultimate weapon for improving sensitivity of gravitational-wave detectors. As we have seen in previous lectures, a GW detector must have a long arm length to have a large displacement signal. However, a long arm gives rise to a delay, which ultimately limits the number of round trips the light can take before GW signal stops to cumulate — there is a trade off between peak sensitivity and bandwidth. Ultimately, the arm length still wins, but that is why SNR only cumulates like $L$, instead of $L^2$.

White-light cavities are cavities that have high finesse but also high bandwidth, thanks to the magic of negative-dispersion devices. These devices provide a frequency dependent phase shift of $\frac{d\Phi(\omega)}{d\omega} < 0$.

As we have such a device, the round trip phase of a signal with sideband frequency $\Omega$ will be given by

$$\frac{\omega_0 L}{c} + \frac{\Omega L}{c} + \left. \frac{d\Phi(\omega)}{d\omega} \right|_{\omega_0} \Omega$$

If we manage to have

$$\frac{d\Phi}{d\omega} = -\frac{L}{c}$$

then the resonant bandwidth of the cavity will be dramatically enhanced with the same input mirror reflectivity. This will have to violate the Mizuno theorem — at least the non-squeezed version. It is the squeezing here that will be really interesting!

3.2.2 Are they too good to be true?

In fact, if we have a device that has $\frac{d\Phi}{d\omega} = -\frac{L}{c}$ and does not have damping of the signal amplitude, it necessarily have the consequence that, a Gaussian wave packet must come out before it comes in — because, as we might recall, $\frac{d\Phi}{d\omega}$ is also the group delay. This is quite interesting!

However, for a band-limited signal, it is OK for the peak to come out before it comes in. As we are given the beginning of the signal, we can analyze it, and deduce its further shape, and produce the peak at the output port before the peak even comes in.

Nevertheless, for a quantum wave packet, we cannot have the same packet come out before it comes in, because we cannot clone a quantum state. This means, what
ever negative-dispersion medium we build, if it does provide an output that comes out before the input, it must be highly damped. If we need it not to be damped, we can amplify it, but with additional noise.

At this moment, we might conclude that white-light cavity is a bad idea — it might look ok for the signal transfer function, but it definitely will create additional noise that will exactly cancel the gain due to the opening of the bandwidth. It is with this belief that Yiqiu Ma, Haixing Miao and I started to write a rigorous proof that white-light cavities should never work for GW detection.

3.3 Proof of the Mizuno Theorem and the Quantum Cramer-Rao Bound

3.3.1 Thought experiment for the Mizuno Theorem

Our approach is to prove the Mizuno Theorem rigorously. In this discussion, we shall ignore quantum mechanical motion of \( x \), but simply consider the shot-noise-limited sensitivity of our detector.
In order to do so, we imagine that the interferometer is at a steady state without signal, then it has a \( \delta(t) \) in displacement. Let us write an interaction Hamiltonian, 

\[ V = -\hbar G_0 (a + a^\dagger) \delta(t) \equiv -\hbar G_0 \sqrt{2} \hat{A} \delta(t) \]  

(3.6)

here \( \hat{A} \) is the amplitude quadrature of the cavity mode. Before and after this impulse, the steady quantum state of the cavity mode will be translated by the signal amplitude, yet its shape unchanged — as shown in the picture. The unitary evolution is given by 

\[ U = \exp[i \sqrt{2} G_0 \hat{A}] \]  

(3.7)

This is a displacement in the phase quadrature 

\[ \hat{P} \equiv \frac{\hat{a} - \hat{a}^\dagger}{\sqrt{2}i} \]  

(3.8)

quadrature by \( \sqrt{2} G_0 \). However, as we detect the out-going field, we can only extract the impulse from the conditional quantum state of the cavity mode — after accounting for all the out-going fields.

That accuracy is no better than the conditional fluctuation of the phase quadrature of the cavity mode — yet that is higher than the inverse of the fluctuations in the amplitude quadrature. In this way, we prove the Mizuno theorem — although we note that the more fluctuations in the amplitude quadrature, the better the SNR.

\[ \int \frac{d\Omega}{2\pi} \frac{1}{S_\mu(\Omega)} \propto E_{AA}, \]  

(3.9)

Now, coming back to the white-light cavity case: the additional noise is indeed going to increase fluctuations in the amplitude quadrature — isn’t this consistent with the Mizuno theorem? In fact, the better question to ask is: how do we know that we will have a conditional pure state — and that we can condition in such a way not to decrease the amplitude-quadrature variance? If we think about it, we can always have a pure state, if we disregard optical losses and thermal noises — and if we account for all the out-going channels. In the case of the negative dispersion medium with additional noise — that just means we have an additional out-going channel, which, after detection and appropriate conditioning, will allow us to go to a pure state for the cavity mode.

To make the above argument rigorous, let us look at the situation with operators at \( t = 0^+ \). We have the incoming field, the out-going field, and the cavity mode operators, basically, we have 

\[ a_{1,2}(x < 0), \quad a_{1,2}(x > 0), \quad \hat{A}, \quad \hat{P} + \sqrt{2} G_0. \]  

(3.10)

Now, if we detect the entire signal from the output port, we will be measuring some linear combination of the above four entities, e.g.

\[ \cos \beta \hat{A} + \sin \beta ( \hat{P} + \sqrt{2} G_0 ) + \int f(x) a(x) dx \]  

(3.11)
Towards a more systematic understanding

This is certainly no better than $\sqrt{2G_0}\sin\beta$ via measuring a conditional fluctuation of $\hat{A}\cos\beta + \hat{P}\sin\beta$. The error is therefore no better than

$$
\epsilon = \frac{V_{AA}\cos^2\beta + 2V_{AP}\sin\beta\cos\beta + \sin^2\beta V_{PP}}{2G_0^2\sin^2\beta}
$$

$$
\geq \frac{1}{2G_0^2} \left[ V_{PP}^\dagger - \frac{V_{AP}V_{AP}^\dagger}{V_{AA}} \right] \geq \frac{1}{V_{AA}} \geq \frac{1}{8G_0^2V_{AA}}
$$

(3.12)

Even though this is a lower bound for the error, it does seem reachable in the case when we do not consider back action.

In this context, the white-light cavity might work, exactly due to the additional noise it imposes. However, we must identify the additional out-going channels that carry away information that we really need in order to complete the conditioning of the cavity mode. Furthermore, the recent proposal by Marquardt’s group, where an internal squeezer in the optical cavity de-amplifies the signal quadrature and amplifies the amplitude quadrature, is right along this approach!

### 3.3.2 The Classical Cramer-Rao Bound

Let us introduce the so-called Quantum Cramer-Rao bound. First of all, there is the classical Cramer-Rao bound, which says that the estimation is related to how the probability distribution changes with respect to the parameter $\theta$.

$$
E(\hat{\theta} - \theta)^2 \leq \frac{1}{I_\theta}
$$

(3.13)

with

$$
I_\theta = E \left[ \left( \frac{\partial \log p(x|\theta)}{\partial \theta} \right)^2 \right] = E \left[ -\frac{\partial^2 \log p(x|\theta)}{\partial \theta^2} \right]
$$

(3.14)

referred to as the Fisher Information. Now suppose we have

$$
x = \theta s(t) + n(t)
$$

(3.15)

then

$$
p(x|\theta) \propto \exp \left[ -\frac{1}{2} \langle x - \theta s|x - \theta s \rangle \right]
$$

(3.16)

with

$$
\langle A|B \rangle \equiv \int \frac{d\Omega}{2\pi} \frac{A^*(\Omega)B(\Omega)}{S_n(\Omega)}
$$

(3.17)

In this way, we have

$$
\frac{\partial^2 \log p(x|\theta)}{\partial \theta^2} = -\langle s|s \rangle
$$

(3.18)

so the bound is given by

$$
E(\hat{\theta} - \theta)^2 \leq \frac{1}{\langle s|s \rangle}
$$

(3.19)
3.3.3 The Quantum Cramer-Rao Bound

Let us write down the quantum version and prove it. Suppose we have a quantum system with density matrix \( \hat{\rho}(\theta) \), which depends on a parameter \( \theta \). We would like to provide an estimator \( \hat{X} \) of \( \theta \), and an unbiased one, with

\[
\text{tr} \left[ \hat{X} \hat{\rho}(\theta) \right] = \theta
\] (3.20)

We will show that there exists a minimum bound for the error we can achieve, which only depends on \( \hat{\rho}(\theta) \).

Let us first define the Logarithmic derivative for \( \hat{\rho} \), namely, we write

\[
\frac{\partial \hat{\rho}}{\partial \theta} = \hat{L} \hat{\rho} - \hat{\rho} \hat{L}
\] (3.21)

where \( \hat{L} \) is a Hermitian operator. (It becomes clear why we define in this way). Then, taking derivative of (3.20), we obtain

\[
\text{tr} \left[ -i \hat{X} \hat{L} \hat{\rho} + i \hat{X} \hat{\rho} \hat{L} \right] = 1
\] (3.22)

We also write

\[
\text{tr} \left[ -i (\hat{X} - \theta) \hat{L} \hat{\rho} + i (\hat{X} - \theta) \hat{\rho} \hat{L} \right] = 1
\] (3.23)

Note that

\[
\text{tr} \left[ -i (\hat{X} - \theta) \hat{L} \hat{\rho} \right] = \left( \text{tr} \left[ i (\hat{X} - \theta) \hat{\rho} \hat{L} \right] \right)^* \] (3.24)

This actually means

\[
\left| \text{tr} [(\hat{X} - \theta) \hat{L} \hat{\rho}] \right|^2 \geq \frac{1}{4}
\] (3.25)

Using the fact that

\[
\text{tr}[A^\dagger A] \text{tr}[B^\dagger B] \geq \left| \text{tr}[A^\dagger B] \right|^2
\] (3.26)

We can write

\[
\text{tr} \left[ \sqrt{\rho}(\hat{X} - \theta)(\hat{X} - \theta) \sqrt{\rho} \right] \text{tr}[\sqrt{\rho}LL\sqrt{\rho}] \geq \left| \text{tr} \left[ \sqrt{\rho}(\hat{X} - \theta)L\sqrt{\rho} \right] \right|^2 \geq 1/4.
\] (3.27)

or

\[
\text{tr}[\rho(\hat{X} - \theta)^2] \geq \frac{1}{4 \text{tr}[\rho L^2]}
\] (3.28)

In our case,

\[
\hat{\rho}_{0+} = U(\theta) \hat{\rho}_0 U(\theta)^\dagger
\] (3.29)

with

\[
U(\theta) = \exp[i\sqrt{2G_0} \theta \hat{A}]
\] (3.30)

In this way, \( L \) is simply given by

\[
L = -\sqrt{2}G_0 \hat{A}
\] (3.31)

This gives an error of

\[
\epsilon \geq \frac{1}{8G_0^2 V_{AA}}
\] (3.32)

As it turned out, Tsang, Wiseman and Caves discussed the Cramer-Rao bound for the sensitivity of force measurement — but we did not realize that it could be applied
here at the time. In fact, Tsang et al. had bound for each frequency $\Omega$ — SNR for each frequency is bounded by the anti-squeezing of the cavity mode at that frequency. In this way, in terms of improving SNR without consideration of optical losses, one must squeeze the quadrature where the signal lives in, and anti-squeeze the amplitude quadrature.

### 3.3.4 Another interpretation of the Cramer-Rao Bound

In the context of linear measurement, we can have another justification why the sensitivity is governed by the fluctuations in the amplitude quadrature — using a reciprocity argument, which was proposed earlier by Yuri Levin, and also discussed by Yiqiu Ma, Bill Kells and David Blair, as well as Peter Saulson. Basically, gravitational wave and optical field are coupled, via an optomechanical and gravitational interaction Hamiltonian. The mode of gravitational wave and the optical field interact, mutually inject information. It is possible to argue that the SNR for detecting the gravity signal is related to the power injected to the gravitational field, due to the gravitational radiation of the detector: the more efficient the detector is, in radiating gravitational wave, the more sensitive it potentially can be — given that the correct out-going field is detected. This argument also extends to the optical-spring case; in fact, Haixing Miao has recently shown that interferometer sensitivity involving the optical spring can also be related to the Cramer-Rao bound, with $V_{AA}$ given by ponderomotive squeezing.

### 3.4 Realizing White-light Cavity using Unstable Filters

Even with white-light cavity in principle possible, we found that the most straightforward implementation involves an unstable filter. For example, Haixing Miao and Yiqiu Ma considered the situation where a cavity is pumped by blue detuned light $\omega_p = \omega_c + \omega_m$, and sending in the carrier at around $\omega_0 = \omega_c$. One has, approximately (when $\gamma_{opt} \gg \gamma_m$)

$$\hat{a}_{out,1,2}(\Omega) \approx \frac{\Omega}{\Omega - i\gamma_{opt}} \hat{a}_{in,1,2}(\Omega)$$

(3.33)

This filter, in the frequency domain, cancels the delay of the arm cavity, and opens up the bandwidth. The fact that the filter is unstable on its own is not a fundamental problem — one can detect out-going field from the entire interferometer, and then use feedback to stabilize the system. This control system does not need to introduce noise — but it makes the frequency-domain sensitivity formula (which has a broad bandwidth and high peak sensitivity) still valid. [Similar scheme can also be realized with Crystal squeezing within a cavity.]

### 3.5 Summary

In this section, we have shown that the gain in sensitivity to gravitational waves can simply be viewed as two steps:

1. For the optical field to couple strongly to the gravitational wave field, we need to have the intra-cavity anti-squeezing of the amplitude as much as possible — this can either be injected squeezing or ponderomotive squeezing.
2. In order to take full advantage of the optomechanical-gravitational coupling, we need to readout all the out-going optical field.
Quantum State Preparation and Verification

I will discuss how a linear quantum measurement device can also be used to prepare and verify the quantum state of the mechanical test masses.

4.1 Zero-Point Fluctuation of an Oscillator and the Fluctuation-Dissipation Theorem

This is already covered by the lecture of A. Clerk. I will not go into the details, but simply mention that a way to look at this is that all oscillators can be viewed to have their zero-point fluctuation driven by external fields.

As we couple the optical cavity to the mass, we now have two baths, the optical one and the mechanical one. The optical field is nearly vacuum, since the frequency of the photon is much higher than $k_B T$. The more we replace the bath by the optical one, the less the temperature is the oscillator; this is *damping* and *cooling*. The optical field can also increase the eigenfrequency of the mechanical oscillator — causing optical dilution.

4.2 Stochastic Schroedinger/Master Equations

Let us mention the Stochastic Master Equation. This is an important equation which finds many applications. For measurement of $x$ with strength $\alpha$, we have

\[
d|\psi\rangle = -\frac{i}{\hbar} \hat{H}|\psi\rangle + \frac{\alpha(\hat{x} - \langle \hat{x} \rangle)}{\sqrt{2}} |\psi\rangle dW - \frac{\alpha^2}{4} (\hat{x} - \langle \hat{x} \rangle)^2 |\psi\rangle dt \tag{4.1}
\]

\[
dy = \alpha\langle \hat{x} \rangle + dW/\sqrt{2} \tag{4.2}
\]

Here $dW$ is the “Wiener increment”; basically $dW/dt$ is a white noise. For a density matrix, we can also replace the first equation above by

\[
d\hat{\rho} = -\frac{i}{\hbar} \left[ \hat{H}, \hat{\rho} \right] + \frac{\alpha}{\sqrt{2}} \{ \hat{x} - \langle \hat{x} \rangle, \rho \} dW - \frac{\alpha^2}{4} [\hat{x}, [\hat{x}, \hat{\rho}]] dt \tag{4.3}
\]

We can remind ourselves how this is derived: during each infinitesimal time interval, we draw some “fresh increment” of independent input field, co-evolve with the system, and then projectively measure the out-going field. In this way, our Hilbert space will be very small: the test mass, plus those of the incoming and out-going field increments.
The third term on the RHS of the SME (4.3) is the Lindblad term. If we throw away the measurement data, we will be left with a master equation with a Lindblad term, which is usually used to describe decoherence. Note that this term is simply caused by an additional force noise that has a white spectrum.

It is important to remember that the SSE/SME are only good at dealing with Markovian systems.

4.3 Wiener Filter and Conditional Gaussian-State Preparation

In this section, we consider detection of the out-going light, conditioning on the measurement result. We shall start with homodyne detection, which corresponds to preparing Gaussian states.

4.3.1 Wiener filters for linear systems at steady state

If we have a linear system, we have analytical solutions to the equations of motion. In this case, we can solve the conditioning problem using a different approach. This will be analogous to linear regression. Let us look at the space-time diagram describing the situation — and we conclude that we just have a linear algebra problem, although with a high number of dimensions. In the situation with conditional state preparation, the out-going fields that arrive up to time $t$, commute with any operator of the test mass.

4.3.2 Pulsed Optomechanics

*Pulsed optomechanics* uses a time-dependent carrier amplitude, which can be experimentally more accessible; it can also be more straightforward to analyze, since we will only need to consider a small number of optical modes.

4.4 Preparation of non-Gaussian Quantum States

4.4.1 Injection of non-Gaussian optical states

Here I discuss using continuous steady pumping. The idea is to inject a single photon from the input port.

4.4.2 Conditioning on non-Gaussian states

4.5 Quantum-State Tomography

Here I discuss how tomography might work.
5

Testing Quantum Mechanics

Let us discuss how to test quantum mechanics. Here one might ask: against what? One can say, we want to test quantum mechanics against classical probability, or we can say we need to perform a precise test of quantum mechanics, with only small deviations from quantum mechanics.

5.1 Collapse Models

Collapse models are constructed for two reasons.

5.2 Semiclassical Gravity

This is a toy model in which we keep gravity to be classical, and then explore how self-gravitating systems can behave.